Toward a Generic Mathematical Model of Abstract Game Theories

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Abstract. Games are a complex mathematical structure for modeling dynamic decision processes under competition where opponent players compete for the maximum gain or toward a success state in the same environment according to the same rules of the game. Games are conventionally dealt with payoff tables based on random strategies, which are found inadequate to describe the dynamic behaviors of games and to rigorously predict the outcomes of games. This paper presents an abstract game theory, which enables a formal treatment of games by a set of mathematical models for both the layouts and behaviors of games. A generic mathematical model of abstract games is introduced, based on which the properties of games in terms of decision strategies and serial matches are described. A wide range of generic zero-sum and nonzero-sum games are formally modeled and analyzed using the generic mathematical models of abstract games.

Keywords: Cognitive informatics, abstract games, game theory, mathematical models, static layout, dynamic behaviors, properties, layoff tables, utilities, decision making, zero-sum games, nonzero-sum games, serial matches, decision grids.

1 Introduction

As a complex mathematical structure, games and game theories are a typical paradigm of decision theories. The study on decision making is interested in multiple disciplines, such as cognitive informatics, computer science, computational intelligence, cognitive psychology, management science, operational theories, economics, sociology, political science, and statistics [5], [10], [14], [17], [20], [24], [25], [30], [33], [34], [35], [36], [38].

Decision theories can be categorized into two paradigms: the *descriptive* and *normative* theories. The former are based on empirical observation and on experimental studies of choice behaviors; and the latter assume a rational decision-maker who

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follows well-defined preferences that obey certain axioms of rational behaviors. Typical normative theories are the expected utility paradigm [12], [17], [21], and the Bayesian theory [5], [33]. W. Edwards and B. Fasolo proposed a 19-step decision making process [10] by integrating Bayesian and multi-attribute utility theories. W. Zachary and his colleagues [42] perceived that there are three constituents in decision making known as the *decision situation*, the *decision maker*, and the *decision process*. Although the cognitive capacities of decision makers may be greatly varying, the core cognitive processes of the human brain share similar and recursive characteristics and mechanisms [35], [39], [40].

An overview of the taxonomy and classification of decision theories and related rational strategies can be illustrated as shown in Fig. 1, which may be used as a guideline for studying the whole framework of decision theories. Most of the decision making strategies [5], [6], [10], [14], [24], [33], [42] can be classified into static decision-making strategies, because the changes of environments of decision makers are independent of the decision makers' activities. Also, different decision strategies may be selected in the same situation or environment based on the decision makers' values and attitudes towards risk and their prediction on future outcomes. In classic decision and operations theories [6], [33], although the states of nature or environment may be both deterministic or nondeterministic, its state of nature as an outcome of the environment will not be changed or affected by the decision maker's actions. In other words, there are natural rules but no adaptive competitors in the static decision making processes.

However, when the environment of a decision maker is interactive with one's decisions or the environment changes according to the decision makers' activities and the decision strategies and rules are predetermined, this category of decision making needs are classified into the category of dynamic decisions, such as games [32], [33] and decision grids [36], [37].

Definition 1. The *dynamic strategies and criteria* of decision making are those that all alternatives and criteria are dependent on both the environment and the effect of the historical decisions made by the decision maker.

Classic dynamic decision making methods are decision trees [6], [12], [17], [21]. A new theory of decision grids is developed in [36], [37] for serial decision makings. Decision making under interactive events and competition is commonly modeled by games [12], [17], [32]. According to Fig. 1, games are used to deal with the most complicated decision problems, which are dynamic, interactive, and under uncontrollable competitions.

Definition 2. A *game* is a decision process under competition where opponent players or opponent groups of players compete for the maximum gain or toward a success state in the same environment according to the same predetermined rules and constraints of the game.

Games traditionally deal with probability-based static payoff tables [3], [4], [12], [16], [19], [26]. However, the conventional approach is found inadequate to deal with the dynamic behaviors of games and to rigorously determine the outcomes of games. This

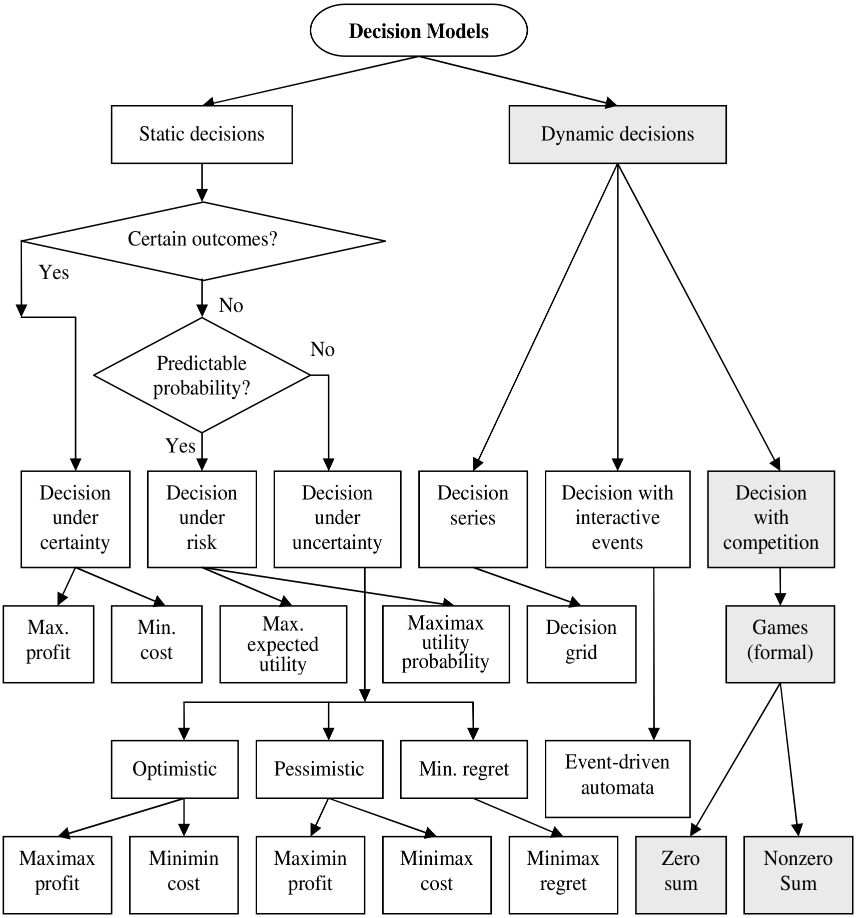

Fig. 1. An overview of decisions and strategies

paper presents a formal model of abstract games, which rigorously describes the architecture or layout of abstract games and their dynamic behaviors with a set of mathematical models. Section 2 reviews related work of various game theories in literature, which indicates a need of a unified model of abstract games. Section 3 develops a formal model of abstract games and describes their properties. Section 4 analyzes the behaviors of abstract games embodied by sets of matches, particularly the zero-sum games and nonzero-sum games. Section 5 discusses strategies of decision making in games such as the maximin and maximum utility strategies.

2 Related Work

Game theory provides a mathematical structure for analyzing the interaction between multiple parties whose decisions affect each other. A game encompasses a finite set of players, a set of courses of actions available to them, and their preferences over the possible outcomes.

Studies on game theories can be traced back to the 1940s when von Neumann and his colleagues studied the theory of *rational games* where the players' decision making pursues best outcomes and maximum utilities under the settings of the game [32]. Von Neumann found that if each player allows the maximin mixed or random strategy, an equilibrium game can be achieved where each strategy is optimal against the other.

John Nash extended von Neumann's rational binary equilibrium to a general case known as *Nash equilibrium* [22], [23]. A set of strategies of an *n*-player game is said to be in *Nash equilibrium* if no player can benefit by deviating from it. Nash proved that there exists at least one equilibrium in games with mixed strategies. However, he did not solve the decision optimization problem when there are multiple equilibriums in a game [8], [13], [15], [28], [41].

Evolutionary games are proposed to enable repeatedly plays of the same game in order to learn best strategies [27], [31]. Kuhn introduced the structure of game trees [18], where the edges represent possible actions and the leaves represent outcomes. In game trees, the players assume perfect information, i.e., any action taken is revealed to all players. Infinitely repeated players are introduced in order to obtain a stable result for a given game under the same rationality constraints [1].

There are also *cooperative* games, where players can form coalitions and make binding agreements about their choice of actions [7], [19], [29], [32]. In cooperative games, if a party of a coalition is treated as a single player for collective decision making, then it can be reduced to a typical noncooperative game [37].

Classic decision theories provide techniques for seeking optimal solutions for a single decision. However, most real world decisions are a process of a series of decisions. This complicated type of serial decisions can be modeled by game theory and decision grids theory. A *decision grid* is a directed network of series decisions over time in which each decision possess only Boolean outcomes, right or wrong, where the effort spent to make a right decision is considered to be identical with that of a wrong decision [36], [37]. The decision grids can be classified into the categories of unlimited and limited grids according to the scope of allowable trials. When the allowable number of trials *t* in a decision grid is infinitive, the decision grid is called an *unlimited decision grid*; otherwise, it is a *limited decision grid*. The unlimited decision grid is a suitable model for the series of decisions toward a success state no matter how many trials are needed, such as an experimental process, a research project, or a person's pursuit towards a goal in life. The limited decision grid is a serial decision model for a short period of trials, such as a student towards a degree, an assessment process, or a deadline-specific process.

3 The Generic Mathematical Model of Abstract Games

Although games are usually represented by layoff tables, the lack of a generic mathematical model of games in game theories has greatly limited the exploration of the modeling, properties, and dynamic behaviors of games. This section develops a mathematical model of a general abstract game, based on which the layouts and behaviors of any concrete game may be treated as a derived instance.

3.1 The Formal Model of Abstract Games

The architecture of an abstract game can be formally described by the following definition, where the behaviors of the game will be modeled by a series of matches between the players of the game.

Definition 3. An *abstract game* G is a 4-tuple, i.e.:

$$G = (P, D, M, S) \tag{1}$$

where

- P is a finite nonempty set of *players* $P = \{p_1, p_2, \ldots, p_n\}$, and n denotes the number of players, $n = \#P,\ n \geq 2$.
- D is a finite nonempty set of *decisions* for certain *moves*, $D = \{d_1, d_2, \ldots, d_k\}$, $k \geq 1$, and all players have the same number of alternative decisions in rational games.
- M is a nonempty finite set of *matches* between players, $M = \{m_1, m_2, \ldots, m_q\}$, $q \geq 1$.
- S is a nonempty finite set of *scores* for each player after a match or a series of matches, $S = \{s_1, s_2, \ldots, s_n\}$.

For a generic abstract game, the matches, which represent the behaviors of the game, can be further described below.

Definition 4. A *match* $m \in M$ of an abstract game $G = (P, D, M, S)$ is a function that maps a set of n decisions made by each player of G into a set of n scores S for each of the players, i.e.:

$$m = f_m : D \times D \times \ldots \times D \to S \tag{2}$$

A match is an individual block given in the payoff table of the game. A set of matches in the given game is constrained by a set of certain rules in order to be rational.

Lemma 1. In an abstract game $G = (P, D, M, S)$, the following rules for matches yield rational, stable, and predictable behaviors and scores:

- **Rule (*a*):** All players are supposed to pursue the maximum gains on the basis of the same predefined payoff table.
- **Rule (*b*):** Whenever the first player initiates a move in a specific set of matches, the remaining moves (actions) of all players in the set of matches are determined according to Rule (a).
- **Rule (*c*):** Each match preset in the payoff table may only be used once in the set of matches.

The rules given in Lemma 1 form the basic constraints of rational games and make a game to be deterministic and its outcomes of all sets of matches are predictable. In lemma 1, Rules (a) and (b) guarantee that all matches of a game are determinable on the basis of the given payoff table. Rule (c) assurances that a set of matches in the game is finite and determinable, although it may force a player to take an unused strategy that would be unfavorable in a particular mach when it is the only strategy left in the setting of the given game.

Lemma 2. The *number of individual matches* n_m in the set of matches of a given game $G = (P, D, M, S)$ is determinable, i.e.:

$$n_m = k^n \tag{3}$$

where n is the number of players in a game, and k is the number of alternative decisions (moves) defined in the game for each player.

Example 1. An $n \times k = 2 \times 2$ game $G_1 = (P, D, M, S)$ can be formally described according to Definition 3 as follows:

- *Players* $P = \{a, b\}$, $n = 2$.
- *Decisions* $D = \{d_1, d_2\}$, $k = 2$, i.e. $D_a = \{a_1, a_2\}$, or $D_b = \{b_1, b_2\}$.
- *Scores* $S = \{s_a, s_b\}$.
- *Matches* $M = \{m_{11}, m_{12}, m_{21}, m_{22}\}$, which is determined by Lemma 2, i.e., $n_m = k^n = 2^2 = 4$.

In $G_1 = (P, D, M, S)$, let a_1 and a_2 be the alternative decisions of player A, and b_1 and b_2 the alternative decisions of player B, then the four matches in M can be formally described as follows:

$$m_{11} = a_1 : b_1 \rightarrow s_a : s_b = 0 : 0$$
$$m_{12} = a_1 : b_2 \rightarrow -1 : 1$$
$$m_{21} = a_2 : b_1 \rightarrow -2 : 2$$
$$m_{22} = a_2 : b_2 \rightarrow 3 : -3$$

The above matches can be represented by a payoff table as shown in Table 1.

Table 1. The Payoff Table of M = $\{m_{11}, m_{12}, m_{21}, m_{22}\}$

	b_1	**b_2**
a_1	0 : 0	-1 : 1
a_2	-2 : 2	3 : -3

This is the static architecture or layout of game G_1. Its dynamic behaviors on the basis of the layout will be discussed in the following subsections.

3.2 Properties of Abstract Games

Properties of games are basic characteristics possessed by them. The properties of abstract games are number of alternative strategies (moves), number of sets of matches, and the number of matches. The properties of abstract games can be used to predicate possible outcomes of games and to select optimal strategies or moves in games.

Definition 5. A *set of matches* in an abstract game $G = (P, D, M, S)$ is a series of matches in which all players may use each of their alternative strategies only once determined according to the current move of opponent and the rule of the maximum gains based on the given layout of the game.

Lemma 3. The *number of set of matches* n_s in an abstract game $G = (P, D, M, S)$ is proportional to both the number of alternative strategies (moves) k, and the number of players n, i.e.:

$$n_s = n \bullet k \tag{4}$$

Lemma 4. The *number of matches* q of an abstract game $G = (P, D, M, S)$ is determined by a product of the number of sets of matches n_s and number of matches in each set n_m, i.e.:

$$\begin{aligned} q &= n_s \bullet n_m \\ &= nk \bullet k^n \\ &= n \bullet k^{n+1} \end{aligned} \tag{5}$$

It is noteworthy that Lemmas 1 through 4 provide a set of generic theories for determining the properties of any given game including those that are beyond the process power of conventional game theories. The attributes of some typical games can be predicated as shown in Table 2.

Table 2. Attributes of Arbitrary and Typical Games

n	$k \Rightarrow (n_m = k^n \mid n_s = n \bullet k \mid q = n_s \bullet n_m = nk^{n+1})$											
	$k = 1$			$k = 2$			$k = 3$			$k = 4$		
2	1	2	2	4	4	16	9	6	54	16	8	128
3	1	3	3	8	6	48	27	9	243	64	12	768
4	1	4	4	16	8	128	81	12	972	256	16	4096
5	1	5	5	32	10	320	243	15	3645	1024	20	20480
…												
100	1	100	100	2^{100}	200	$100 \bullet 2^{101}$	3^{100}	200	$100 \bullet 3^{101}$	4^{100}	400	$100 \bullet 4^{101}$

According to Table 2, the complexity of games is explosively increasing proportional to the numbers of both players n and strategies k. This explains why games are so complicated and difficult to be modeled and formally treated in conventional game theory [6], [32]. For example, when the number of players $n = 5$ and the number of alternative strategies of each player $k = 4$, the total number of matches of the game may easily reach as high as 20,480. That is why conventional empirical game theories may only deal with small and simple games with a few of players and alternative strategies.

However, the abstract game theory presented so far is able to analyze any games no matter how large n and k would be based on the generic mathematical model of formal games and their instances.

Since games with multiple players can be divided into a number of pairwise games, the following sections will focus on the analyses of binary game properties as shaded in Table 2.

Definition 6. A *binary game* $G = (P_2, D, M, S_2)$ is a game with only two players $n = 2$, where $P_2 = \{p_1, p_2\}$ and $S_2 = \{s_1, s_2\}$, simply called a game.

Example 2. A well known binary game is the Prisoner's dilemma, where two conspirators in prison may receive a sentence for either two years or eight years for both remaining silent or confess, respectively. They are also given the opportunity to confess in return for a reduced prison sentence of half a year. The payoffs correspond to numbers of years in prison are given in Table 3.

Table 3. The Payoff Table of the Prisoner's Dilemma

	$\mathbf{b_1}$ (silent)	$\mathbf{b_2}$ (confess)
$\mathbf{a_1}$ (silent)	-2 : -2	-0.5 : -10
$\mathbf{a_2}$ (confess)	-10 : -0.5	-8 : -8

It is noteworthy that according to Nash equilibrium, the utility of the above game is (-8, -8), rather than (-2,-2) [2], [22]. However, according to the abstract game theory, the average score of the above game is -20.5 : -20.5, i.e., there is no winner according to the payoff table of the Prisoner's dilemma.

When a game $G = (P, D, M, S)$ is set according to Definitions 3 and 4, the properties of G, such as the number of matches, the number of sets of matches, and the winner are determined. Game theory may be used to predict and select the optimal combination of individual strategies. However, the score for any individual strategy in G has already fixed according to the payoff table.

Theorem 1. The *properties of games* state that an abstract game $G = (P, D, M, S)$ is *deterministic* and *conservative*. Once the game G is set, the properties of G are determined, predictable, and unchangeable to all players in the game.

According to Theorem 1, game theory may be used to predict and select the optimal combinations of individual strategies for a player in a given game G. However, the optimal strategies may not necessarily result in a win situation rather than a minimal loss in some cases, because the scores for individual moves and their combination strategies in G are determined by the settings of the game.

Corollary 1. The outcomes of a formal game $G = (P, D, M, S)$ are constrained by the settings of the game. Although an individual strategy may result in the maximum gain, the final score of a player in the whole set of games is fixed by the payoff table in a particular match, which may not necessarily result in a win situation for all players.

The objective of decision makers in a game is to make the score of a player to the maximum. However, according to Corollary 1, $\max(s_i)$ may not mean a winning score due to the settings of a given game.

4 Behaviors of Abstract Games

There are zero-sum and nonzero-sum games. Each of them has different properties and dynamic behaviors as described in the following subsections.

4.1 Behaviors of Zero-Sum Games

Definition 7. A *zero-sum game* is a type of abstract games where the total score of all players in the game remains zero, i.e.:

$$\sum_{i=1}^{n} s_i = 0 \tag{6}$$

In the case of a binary game, Eq. 6 can be expressed as follows:

$$s_1 = - s_2 \tag{7}$$

where Eq. 7 models a decision making situation that one player's gain is always another's loss.

Lemma 5. The *condition for a zero-sum game* is that all n_m individual matches are zero-sum, i.e.:

$$\sum_{i=1}^{n_m} s_i = 0 \tag{8}$$

Example 3. The game $G_1 = (P, D, M, S)$ as given in Example 1 and Table 1 is a zero-sum game. The properties and behaviors of G_1 can be formally analyzed below.

The properties of $G_1 = (P, D, M, S)$ are:

- *Number of sets of matches:* $n_s = n \bullet k = 2 \bullet 2 = 4$
- *Number of matches in a set:* $n_m = k^n = 2^2 = 4$
- *Total number of matches in the game:*
  $n_m = n_s \bullet n_m = n \bullet k^{n+1} = 2 \bullet 2^3 = 16$

The four sets of matches each with a series of four individual matches can be illustrated in Fig. 2.

$$
\begin{array}{ll}
 & s_a : s_b \\
\text{Set 1: } a_1 \xrightarrow{-1:1} b_2 \xrightarrow{3:-3} a_2 \xrightarrow{-2:2} b_1 \xrightarrow{0:0} a_1 & \Rightarrow 0 : 0 \\
\text{Set 2: } a_2 \xrightarrow{-2:2} b_1 \xrightarrow{0:0} a_1 \xrightarrow{-1:1} b_2 \xrightarrow{3:-3} a_2 & \Rightarrow 0 : 0 \\
\text{Set 3: } b_1 \xrightarrow{0:0} a_1 \xrightarrow{-1:1} b_2 \xrightarrow{3:-3} a_2 \xrightarrow{-2:2} b_1 & \Rightarrow 0 : 0 \\
\text{Set 4: } b_2 \xrightarrow{3:-3} a_2 \xrightarrow{-2:2} b_1 \xrightarrow{0:0} a_1 \xrightarrow{-1:1} b_2 & \Rightarrow 0 : 0
\end{array}
$$

Fig. 2. Sets of matches in the zero-sum game G_1

Lemma 6. The *final scores of all sets of matches* of an abstract games *G* are the same, no matter who moves first and which strategy (decision alternative) is selected for the first move.

Theorem 2. The *scores* of a $2 \times k$ abstract game, $s_a : s_b$, is predetermined by the settings of the payoff table, i.e.:

$$
\begin{aligned}
s_a : s_b &= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{a}\right) : \left(\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{b}\right) \\
&= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{a}\right) : \left(-\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{a}\right)
\end{aligned} \tag{9}
$$

where *k* is the number of alternative decision strategies and *k* is identical for all players.

According to Theorem 2, the results of all possible sets of matches for a given zero-sum game can be predicated using Eq. 9. For instance, the final score of Example 3 can be calculated according Eq. 9 as follows:

$$
\begin{aligned}
s_a : s_b &= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{a}\right) : \left(\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^{b}\right) \\
&= (s_{11}^{a} + s_{12}^{a} + s_{21}^{a} + s_{22}^{a}) : \\
&\quad (s_{11}^{b} + s_{12}^{b} + s_{21}^{b} + s_{22}^{b}) \\
&= (0 - 1 - 2 + 3) : \\
&\quad (0 + 1 + 2 - 3) \\
&= 0 : 0
\end{aligned}
$$

Example 4. For a 2×3 game $G_2 = (P, D, M, S)$ with the following payoff table, try to determine its properties and behaviors.

Table 4. The Payoff Table of $G_2 = (P, D, M, S)$

	b₁	**b₂**	**b₃**
A₁	0 : 0	100 : -100	200 : -200
A₂	-300 : 300	0 : 0	-100 : 100
A₃	500 : -500	-200 : 200	0 : 0

The properties of $G_2 = (P, D, M, S)$ are:

- *Number of sets of matches:* $n_s = n \bullet k = 2 \bullet 3 = 6$
- *Number of matches in a set:* $n_m = k^n = 3^2 = 9$
- *Total number of matches in the game:*
  $n_m = n_s \bullet n_m = n \bullet k^{n+1} = 2 \bullet 3^3 = 54$

According to Theorem 2, the final scores of $G_2 = (P, D, M, S)$ are as follows:

$$
\begin{aligned}
s_a : s_b &= (\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^a) : (\sum_{i=1}^{k}\sum_{j=1}^{k} s_{ij}^b) \\
&= (s_{11}^a + s_{12}^a + s_{13}^a + s_{21}^a + s_{22}^a + s_{23}^a + s_{31}^a + s_{32}^a + s_{33}^a) : \\
&\quad (s_{11}^b + s_{12}^b + s_{13}^b + s_{21}^b + s_{22}^b + s_{23}^b + s_{31}^b + s_{32}^b + s_{33}^b) \\
&= (0 + 100 + 200 - 300 + 0 - 100 + 500 - 200 + 0) : \\
&\quad (0 - 100 - 200 + 300 + 0 + 100 - 500 + 200 + 0) \\
&= 200 : \text{-}200
\end{aligned}
$$

The behaviors of $G_2 = (P, D, M, S)$ can be modeled by 54 detailed matches in 6 sets as shown in Fig. 3.

Corollary 2. The *outcomes* of a given game G is determined according to the scores $s_a : s_b$ as follows:

$$
\begin{cases}
s_a > s_b\text{: Player A won} \\
s_a = s_b\text{: Tie} \\
s_a < s_b\text{: Player B won}
\end{cases}
\tag{10}
$$

Therefore, the final score of Example 1, $s_a : s_b = 0 : 0$, shows a tie game; while Example 3, $s_a : s_b = 200 : \text{-}200$, indicates that Player A will always win.

4.2 Behaviors of Nonzero-Sum Games

A more general type of games is nonzero-sum games where all players involved share a certain pie with a fixed size. From this view, the zero-sum game discussed in Section 4.1 is a special case of nonzero-sum games where the size of the pie is zero.

$$
\begin{array}{ll}
 & s_a : s_b \\
\text{Set 1: } a_1 \xrightarrow{0:0} b_1 \xrightarrow{500:-500} a_3 \xrightarrow{-200:200} b_2 \xrightarrow{100:-100} a_1 & \\
\quad \xrightarrow{200:-200} b_3 \xrightarrow{0:0} a_3,\ a_2 \xrightarrow{-300:300} b_1, & \\
\quad a_2 \xrightarrow{0:0} b_2,\ a_2 \xrightarrow{-100:100} b_3 & \Rightarrow 200 : -200 \\
\text{Set 2: } a_2 \xrightarrow{-300:300} b_1 \xrightarrow{500:-500} a_3 \xrightarrow{-200:200} b_2 \xrightarrow{100:-100} a_1 & \\
\quad \xrightarrow{0:0} b_1, a_1 \xrightarrow{200:-200} b_3,\ a_2 \xrightarrow{0:0} b_2, & \\
\quad a_2 \xrightarrow{-100:100} b_3,\ a_3 \xrightarrow{0:0} b_3 & \Rightarrow 200 : -200 \\
\text{Set 3: } a_3 \xrightarrow{-200:200} b_2 \xrightarrow{100:-100} a_1 \xrightarrow{0:0} b_1 \xrightarrow{500:-500} a_3 & \\
\quad \xrightarrow{0:0} b_3 \xrightarrow{200:-200} a_1,\ a_2 \xrightarrow{-300:300} b_1, & \\
\quad a_2 \xrightarrow{0:0} b_2,\ a_2 \xrightarrow{-100:100} b_3 & \Rightarrow 200 : -200 \\
\text{Set 4: } b_1 \xrightarrow{500:-500} a_3 \xrightarrow{-200:200} b_2 \xrightarrow{100:-100} a_1 \xrightarrow{0:0} b_1 & \\
\quad \xrightarrow{-300:300} a_2 \xrightarrow{-100:100} b_3,\ b_2 \xrightarrow{0:0} a_2, & \\
\quad b_3 \xrightarrow{200:-200} a_1,\ b_3 \xrightarrow{0:0} a_3 & \Rightarrow 200 : -200 \\
\text{Set 5: } b_2 \xrightarrow{100:-100} a_1 \xrightarrow{0:0} b_1 \xrightarrow{500:-500} a_3 \xrightarrow{-200:200} b_2 & \\
\quad \xrightarrow{0:0} a_2 \xrightarrow{-300:300} b_1,\ b_3 \xrightarrow{200:-200} a_1, & \\
\quad b_3 \xrightarrow{-100:100} a_2,\ b_3 \xrightarrow{0:0} a_3 & \Rightarrow 200 : -200 \\
\text{Set 6: } b_3 \xrightarrow{200:-200} a_1 \xrightarrow{0:0} b_1 \xrightarrow{500:-500} a_3 \xrightarrow{-200:200} b_2 & \\
\quad \xrightarrow{100:-100} a_1, a_3 \xrightarrow{0:0} b_3,\ b_1 \xrightarrow{-300:300} a_2, & \\
\quad b_2 \xrightarrow{0:0} a_2,\ b_3 \xrightarrow{-100:100} a_2 & \Rightarrow 200 : -200
\end{array}
$$

Fig. 3. Sets of matches of the 2×3 zero-sum game

Definition 8. A *nonzero-sum game* is a game where the total scores of all players in the game is a nonzero positive value, i.e.:

$$\sum_{i=1}^{n} s_i > 0 \tag{11}$$

A group on a common project or a set of partners bidding for a contract is a typical example of nonzero-sum games.

The most interesting property of decision making in nonzero-sum game is that there is an ideal state of result known as the win-win situation.

Definition 9. A *win-win game* is a nonzero-sum game in which all players gain a certain score constrained by Eq. 11.

Lemma 7. A win-win game can only exist in nonzero-sum games.

According to Lemma 7, if a number of competitive players in a nonzero-sum game are coordinated, i.e., a superset of partnership is established in the game, every party may be benefit.

Theorem 3. A *win-win decision* can be achieved in a nonzero-sum game when the following condition is satisfied:

$$\sigma \geq \frac{1}{n_s}\sum_{i=1}^{n_s} s_i \tag{12}$$

where σ is the sum of the game that is a nonzero positive constant, s_i is the expected score of player *i*, and n_s is the number of sets of matches in the game.

According to Theorem 3, a win-win game may satisfy all coordinative players when the constant sum σ is large enough as determined by Eq. 12.

Example 5. Given a 2 × 2 nonzero-sum game $G_3 = (P, D, M, S)$ with the following payoff table and σ = 100, try to determine its properties and behaviors.

Table 5. The Payoff Table of $G_3 = (P, D, M, S)$

	b₁	**b₂**
a₁	70 : 30	20 : 80
a₂	60 : 40	90 : 10

The properties of $G_3 = (P, D, M, S)$ are:

- *Number of sets of matches:* $n_s = n \bullet k = 2 \bullet 2 = 4$
- *Number of matches in a set:* $n_m = k^n = 2^2 = 4$
- *Total number of matches in the game:*
  $n_m = n_s \bullet n_m = n \bullet k^{n+1} = 2 \bullet 2^3 = 16$

According to Theorem 2, the final scores of $G_3 = (P, D, M, S)$ are as follows:

$$\begin{aligned} s_a : s_b &= (\sum_{i=1}^{k}\sum_{j=1}^{k} s^a_{ij}) : (\sum_{i=1}^{k}\sum_{j=1}^{k} s^b_{ij}) \\ &= (s^a_{11} + s^a_{12} + s^a_{21} + s^a_{22}) : (s^b_{11} + s^b_{12} + s^b_{21} + s^b_{22}) \\ &= (70 + 20 + 60 + 90) : (30 + 80 + 40 + 10) \\ &= 240 : 160 \end{aligned}$$

This result indicates that the four sets of matches defined by G_3 will result in an average score in each match as 60 : 40, in which Players A and B share σ = 100. This can be proved by the following four sets of matches as shown in Fig. 4.

	$s_a : s_b$
Set 1: $a_1 \xrightarrow{20:80} b_2 \xrightarrow{90:10} a_2 \xrightarrow{60:40} b_1 \xrightarrow{70:30} a_1$	$\Rightarrow 240 : 160$
Set 2: $a_2 \xrightarrow{60:40} b_1 \xrightarrow{70:30} a_1 \xrightarrow{20:80} b_2 \xrightarrow{90:10} a_2$	$\Rightarrow 240 : 160$
Set 3: $b_1 \xrightarrow{70:30} a_1 \xrightarrow{20:80} b_2 \xrightarrow{90:10} a_2 \xrightarrow{60:40} b_1$	$\Rightarrow 240 : 160$
Set 4: $b_2 \xrightarrow{90:10} a_2 \xrightarrow{60:40} b_1 \xrightarrow{70:30} a_1 \xrightarrow{20:80} b_2$	$\Rightarrow 240 : 160$

Fig. 4. Sets of matches of the 2×2 nonzero-sum game G_3

It may be observed that for a given game in a certain context, it would appear to be competitive between conflict interests of players. However, at a higher level of an enlarged scope of the given game, it can be perceived differently as a cooperative game for all parties involved. This leads to the following corollary for management attitude and skills in decision making.

Corollary 3. The *art of management*, to a certain extent, is to create a win-win environment for all members, partners, and parent organizations involved in a game context.

5 Strategies of Decision Making in Games

According to Lemma 4, a certain layout of a game implies a whole set of $q = nk^{n+1}$ match outcomes. Therefore, the choice of decision strategies in games is crucial. The following subsections discuss typical decision making strategies in games known as those of maximin and maximum utility.

5.1 The Maximin Strategy in Games

A conservative strategy to try to gain the maximum utility or to spend the minimum cost under uncertainty is known as the maximin strategy in games [9], [37].

Definition 10. A *conservative decision making under uncertainty* $d_{maximin}$ or $d_{minimax}$ yields a decision with the *maximum-minimum* strategy for utility or a *minimum-maximum* strategy for cost, i.e.:

$$\begin{aligned} d_{maximin} &= f\text{: } \mathcal{A} \times C \rightarrow \mathcal{A} \\ &= \{a_i \mid max\ (min\ (u_{ij} \mid 1 \le i \le n) \mid 1 \le j \le k)\} \end{aligned} \tag{13a}$$

or

$$\begin{aligned} d_{minimax} &= f\text{: } \mathcal{A} \times C \rightarrow \mathcal{A} \\ &= \{a_i \mid min\ (max\ (u_{ij} \mid 1 \le i \le n) \mid 1 \le j \le k)\} \end{aligned} \tag{13b}$$

where $\mathcal{A}$ is a set of given alternative strategies, and C a set of decision-making criteria.

Example 6. Consider the following engineering project where a maximin or a pessimistic uncertainty decision can be made based on the project gains for different architecture-result combinations as shown in Table 6.

Table 6. Maximin Decision Making for an Engineering Project

Alternative ($\mathcal{A}$)	Situation (S)				Criterion (maximin utility)
	Result 1 (s_1)	Result 2 (s_2)	Result 3 (s_3)	Result 4 (s_4)	
Architecture (a_1)	100	10	40	60	u_{12} = \$10k
Architecture (a_2)	- 10	50	200	30	
Architecture (a_3)	50	20	5	130	

According to Eq. 13a, the maximin decision under uncertainty is as follows:

$$\begin{aligned} d_{maximax} &= f\colon \mathcal{A} \times C \to \mathcal{A} \\ &= \{a_i \mid max\ (min\ (u_{ij} \mid 1 \leq i \leq 3) \mid 1 \leq j \leq 4)\} \\ &= \{a_i \mid max\ (u_{12}, u_{21}, u_{33})\} \\ &= \{a_1 \mid u_{12} = 10\} \end{aligned}$$

The solution indicates that the conservative decision for this given project with the maximin criterion is (a_1, s_2), which will result in a maximin project gain $u_{max} = u_{12} =$ \$10,000. It is noteworthy that, by choosing this solution, there is a chance to lose the opportunity gain of u_{23} =\$200,000 if the uncertain outcomes turn out to be Result 3 constrained by the other party of the game. However, in any case, this decision can prevent the project from a negative result as that of u_{21} = -\$10,000.

5.2 The Maximum Utility Strategy in Games

Subsection 5.1 deals with decision strategies where the probabilities of the opponent party are uncertain. When the opponent strategies in a game are individually predictable, i.e., the probabilities or likelihoods are known, the risk for a decision can be better estimated. In this case, decision making process will be directed based on the weights of probabilities for each payoff.

Definition 11. A *decision making under risk* is a selection of an alternative a_i among $\mathcal{A}$ that meets a given criterion C, when the likelihood or probability of each possible situation is known or can be predicated.

The criterion for a decision making under risk can be based on the maximum expected utility of alternatives.

Definition 12. An *expected utility EU* is a weighted sum of all utilities u_j for each decision alternative based on known probabilities for each possible situation p_j, i.e.:

$$EU_i = \sum_{j=1}^{k} u_{ij} \bullet p_j, \ 1 \le i \le n \tag{14}$$

Definition 13. A *decision making under risk with maximum expected utility* d_{maxEU} yields a decision with the *maximum expected utilities* of all alternatives, i.e.:

$$\begin{aligned} d_{maxEU} &= f\!: \mathcal{A} \times C \to \mathcal{A} \\ &= \{a_i \mid max\ (EU_i \mid 1 \le i \le n)\} \end{aligned} \tag{15}$$

Example 7. Consider the same layout given in Example 6. A decision under risk with maximum expected utility can be made based on the *EUs* determined by Eq. 14 for different decision alternatives as shown in Table 7.

After the expected utilities for all three alternatives are obtained as shown in Table 7, the best decision with the maximum expected utility can be determined according to Eq. 15 as follows:

$$\begin{aligned} d_{maxEU} &= f\!: \mathcal{A} \times C \to \mathcal{A} \\ &= \{a_i \mid max\ (EU_i \mid 1 \le i \le n)\} \\ &= \{a_i \mid man\ (EU_1, EU_2, EU_3)\} \\ &= \{a_2 \mid EU_2 = 66\} \end{aligned}$$

Table 7. Decision Making based on the Maximum Expected Utility for an Engineering Project

Alternative ($\mathcal{A}$)	**Situation (*S*)**				**Expected Utility (EU)**	**Criterion (Maximum EU)**
	Result 1 (s_1) [p_1 = 0.2]	**Result 2 (s_2)** [p_2 = 0.5]	**Result 3 (s_3)** [p_3 = 0.2]	**Result 4 (s_4)** [p_4 = 0.1]		
Architecture (a_1)	100	10	40	60	$EU_1 = 39$	
Architecture (a_2)	- 10	50	200	30	$EU_2 = 66$	$EU_{max} = 66$
Architecture (a_3)	50	20	5	130	$EU_3 = 34$	

The solution indicates that the decision under risk for this given project with the maximum expected utility criterion is Architecture a_2 that will result in a maximum weighted sum EU_2 = $66,000.

Decision making under risk with the maximum expected utility d_{maxEU} can be described by a backward-inducted *decision tree* as shown in Fig. 5. The decision tree provides another approach to derive the maximum expected utility in two steps [11]. First, the individual weighted utilities of all the alternatives are calculated according to Eq. 14, which yields EU_i, $1 \le i \le 3$, represented by the three middle nodes. Then, the maximum utility EU_{max} is selected from these three middle nodes according to Eq. 15, which yields node *A* represented by decision d_2 with $EU_{max} = 66$.

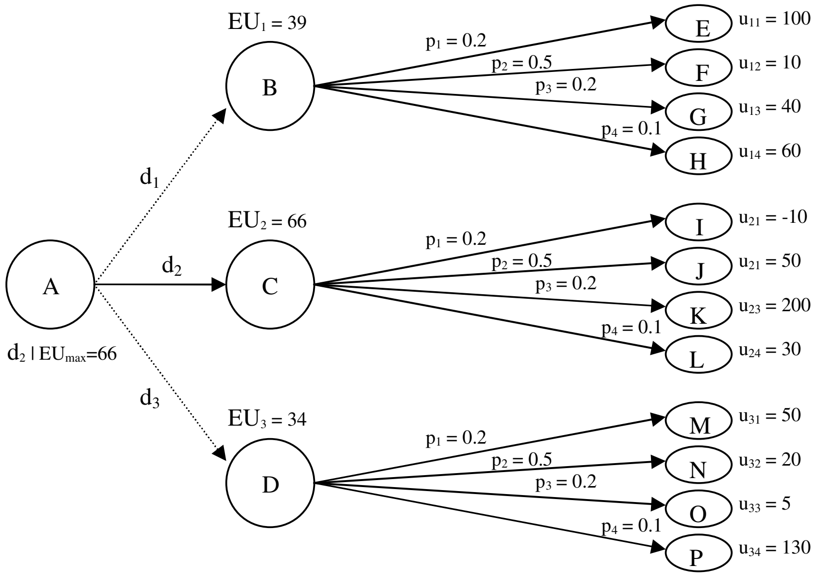

Fig. 5. A decision tree based on the strategy of maximum expected utility

6 Conclusions

This paper has presented a generic mathematical model of abstract games. Based on the abstract game theory, properties of games have been analyzed, and the predictability of games has been derived. This paper has demonstrated a formal treatment of games by a set of mathematical models on both of the layout and behaviors of abstract games in terms of serial matches constrained by the generic rules. Then, all specific games has been treated as particular instances. On the basis of the generic game theory, zero-sum and nonzero-sum games have been rigorously analyzed and their properties and relations have been formally described. A set of decision strategies of games such as the maximin and maximum utility has been explained. The generic abstract game theories and mathematical models can be applied in the design and implementation of game systems as well as autonomous agent systems.

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